# ON THE DYNAMIC STABILITY OF THIN ELASTIC SHELLS FILLED WITH A LIQUID

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We shall study in the following the dynamic stability of a thin elastic shell filled partly or completely with an ideal incompressible liquid. Use will be made of a paper by Moiseev [1] which deals with free vibrations of an elastic beam containing a liquid, as well as of [2] and [3], devoted to the study of dynamic and static stability of elastic systems without a liquid.

**1.** The problem indicated reduces to the solution of the variational equation

$$\delta \int_{t_0}^{t} (T'' - A'' - U'')\, dt = 0 \qquad (1.1)$$

where $T''$ and $U''$ are the kinetic and the potential energies, respectively, of the perturbed system, so that

$$T'' = \frac{1}{2}\left\{\iint_{\Sigma} m_0 \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] d\sigma + \rho \iint_{\Sigma + S} \varphi \frac{\partial \varphi}{\partial n}\, d\sigma\right\} \qquad (1.2)$$

$$U'' = \frac{1}{2}\iint_{\Sigma} [T_1\varepsilon_1 + T_2\varepsilon_2 + 2S\omega - M_1\varkappa_1 - M_2\varkappa_2 - 2H\tau]\, d\sigma + \iint_{S} \left(\frac{\partial \varphi}{\partial t}\right)^2 d\sigma \qquad (1.3)$$

while the quantity $A''$ represents the work of a certain reduced loading, arising as a result of perturbations of the system, done in the displacements of such perturbations. The determination of the reduced loading is achieved in the same manner as in [2]. In such cases when, in the determination of these loadings, the inertial terms can be neglected, or when the initial state (the state which is to be investigated for stability) of the shell is approximately a membrane state, they will be

$$F_\beta = \frac{1}{PQ}\left[\frac{\partial}{\partial\beta}(\varepsilon_1 P T_2^0) - T_1^0 \frac{\partial}{\partial\beta}(\varepsilon, P) + \frac{\partial}{\partial\alpha}(\varepsilon_2 P S^\circ) + S^\circ \frac{\partial}{\partial\alpha}(\varepsilon_2 Q) - q_\beta(\varepsilon_1 + \varepsilon_2)\right]$$
$$F_n = T_1^\circ \varkappa_1 + T_2^\circ \varkappa_2 \tag{1.4}$$
$$F_\alpha = \frac{1}{PQ}\left[\frac{\partial}{\partial\alpha}(\varepsilon_2 Q T_1^\circ) - T_2^\circ \frac{\partial}{\partial\alpha}(\varepsilon_2 Q) + \frac{\partial}{\partial\beta}(\varepsilon_1 Q S^\circ) + S^\circ \frac{\partial}{\partial\beta}(\varepsilon_1 p) - q_\alpha(\varepsilon_1 + \varepsilon_2)\right]$$

and then

$$A'' = \frac{1}{2}\iint_\Sigma [F_\alpha u + F_\beta v + F_n w]\, d\sigma \tag{1.5}$$

Here and in the following we shall denote by: $\Sigma$ the middle surface of the shell; $\alpha$, $\beta$ the curvilinear orthogonal coordinates in the middle surface; $n$ a normal to the middle surface; $P$, $Q$ the coefficients of the first quadratic form of the middle surface; $u$, $v$, $w$ the displacement components of a perturbation of the shell in the directions of $\alpha$, $\beta$, $n$, respectively; $m_0$, $\rho$ mass densities of unit surface area of the shell and unit volume of the liquid, respectively; $\epsilon_1$, $\epsilon_2$, $\omega_1\kappa_1$, $\kappa_2$, $r$ relative deformations of the shell, expressible in terms of $u$, $v$, $w$ by means of well-known formulas of the linear theory of shells; $T_1$, $T_2$, $S$, $M_1$, $M_2$, $H$ stress resultants and couples of the shell, expressible in terms of the relative deformations by means of Hooke's law; $T_1^\circ$, $T_2^\circ$, $S^\circ$ stress resultants of the non-perturbed shell, which characterize the initial membrane state of the shell; $q_\alpha$, $q$, $q_n$ external surface loadings, acting on the shell, possibly functions of time; $a$ the acceleration of the translational motion of the system; $S$ the free surface of the liquid in the state of rest; $V$ the volume occupied by the liquid; $\phi$ velocity potential of the liquid in that domain; $\Sigma^{(1)}$ the wetted surface of the shell; $\Sigma_1$ the part of the boundary of $V$ where the derivative $\partial\rho/\partial h$ is known; $\Sigma_2$ the part of the boundary of $V$ where the function $\phi$ is known; $G$ Green's function of Neumann-Dirichlet's problem for Laplace's equation in the domain $V$.

The potential $\phi$ can be expressed in terms of $G$ in the following manner:

$$\varphi = \iint_{\Sigma_1} G\frac{\partial\varphi}{\partial n}\, d\sigma - \iint_{\Sigma_2}\frac{\partial G}{\partial n}\varphi\, d\sigma \tag{1.6}$$

The solution of Equation (1.1) reduces to the solution of the four differential equations

$$L_{11}(u) + L_{12}(v) + L_{13}(w) + \frac{1-\nu^2}{Eh}\left[F_\alpha - m_0\frac{\partial^2 u}{\partial t^2}\right] = 0$$

$$L_{21}(u)+L_{22}(v)+L_{23}(w)+\frac{1-\nu^2}{Eh}\left[F_\beta-m_0\frac{\partial^2 v}{\partial t^2}\right]=0$$
$$L_{31}(u)+L_{32}(v)+L_{33}(w)+\frac{1-\nu^2}{Eh}\left[F_n-m_0\frac{\partial^2 w}{\partial t}-\rho\frac{\partial\varphi}{\partial t}\right]=0 \tag{1.7}$$
$$\Delta\varphi=0$$

with boundary conditions corresponding to fixed edges of the shell, and conditions for the velocity potential of the liquid on the boundary of the domain $V$. The edges of the shell can be fixed in various ways, therefore we do not formulate here the boundary conditions corresponding to them explicitly. The conditions for $\phi$ are

$$\frac{\partial^2\varphi}{\partial t^2}+a\frac{\partial\varphi}{\partial z}=0 \quad \text{on the free surface } z=0 \tag{1.8}$$
$$\frac{\partial\varphi}{\partial n}=\frac{\partial w}{\partial t} \quad \text{on the wetted middle surface of the shell} \tag{1.9}$$

The operators $L_{11}, \ldots, L_{33}$ appearing in Equations (1.7) are the left-hand members of the equations of the theory of shells [4,5].

2. It is not difficult to introduce into the discussion such operators $L$, $M$, $E$, $N$ and a vector $X(u, v, w, \phi)$, so that Equations (1.7) appear in the form

$$LX+MX+E\frac{\partial^2 X}{\partial t^2}+N\frac{\partial X}{\partial t}=0 \tag{2.1}$$

The operators $L$, $M$, $E$, $N$ satisfy the conditions of existence and uniqueness of the generalized solution in accordance with Theorem 3 of [6].

3. As an application of the theory developed above we consider in the following the question of dynamic stability of a circular cylindrical shell filled with a liquid and with hinged edges.

Let us introduce a system of cylindrical coordinates $r$, $z$, $\theta$ oriented in the usual way. Assume the cylinder under consideration of radius $R$ to be situated between the planes $z = 0$ and $z = l$. The inner space of the cylinder is partly filled with a liquid of density $\rho$ up to a level $z = l_1$. We choose the curvilinear coordinates $a$ and $\beta$ in such a manner as to have $\beta = \theta$, $a = z/R$, where $\theta$ is the angular coordinate of the cylindrical system.

Assume the cylindrical shell considered to be under the action of a distributed radial loading $q_n(a, t)$. This can be, for example, a hydrostatic loading. The shell is, in addition, acted upon by a longitudinally distributed load $q_\alpha(\alpha, t)$ and by a longitudinal compressive force $F(t)$.

For the case under consideration the coefficients of the first quadratic form will be $P = Q = R$, while the curvatures are $K_1 = 0$, $K_2 = 1/R$.

We assume that the shell is initially in a membrane state of stress, characterized by the stress resultants $T_1^\circ$, $T_2^\circ$.

Integrating the equations of the cylindrical shell in the membrane state of stress we obtain

$$T_1^\circ = R\int_0^\alpha q_\alpha(\alpha, t)\,d\alpha + \frac{1}{2\pi R}F(t), \qquad T_2^\circ = Rq_n(\alpha, t) \tag{3.1}$$

The relative deformations of the cylindrical shell will be

$$\varepsilon_1 = \frac{1}{R}\frac{\partial u}{\partial \alpha}, \qquad \varepsilon_2 = \frac{1}{R}\left(\frac{\partial v}{\partial \beta} + w\right), \qquad \varkappa_1 = \frac{1}{R^2}\frac{\partial^2 w}{\partial \alpha^2}, \qquad \varkappa_2 = -\frac{1}{R^2}\left(\frac{\partial^2 w}{\partial \beta^2} + w\right) \tag{3.2}$$

The components of the reduced loading are determined by Formulas (1.4), which lead to

$$F_\alpha = \frac{1}{R^2}(T_1^\circ - T_2^\circ)\frac{\partial}{\partial \alpha}\left(\frac{\partial v}{\partial \beta} + w\right) - \frac{1}{R^2}\frac{\partial u}{\partial \alpha}\frac{\partial^2 T_1^\circ}{\partial \alpha^2}$$

$$F_\beta = \frac{1}{R^2}(T_2^\circ - T_1^\circ)\frac{\partial^2 u}{\partial \alpha\,\partial \beta}, \qquad F_n = -\frac{1}{R^2}\left\{T_1^\circ\frac{\partial^2 w}{\partial \alpha^2} + T_2^\circ\left(\frac{\partial^2 w}{\partial \beta^2} + w\right)\right\} \tag{3.3}$$

We now turn to the determination of the velocity potential $\phi$ of the motion of the liquid.

Relaxing the boundary conditions at the section $z = 0$ and neglecting the wave motion on the free surface, we shall assume that at the sections $z = 0$ and $z = l_1$ we know the additional pressure $p$ produced by the perturbation of the system under consideration.

It is obvious that such an assumption is justified in the case of long cylinders, because in this case the energy of the wave motion is small as compared with the total energy of the system and, besides, long cylinders lose stability in a non-axisymmetrical deformation, when

$$\iint_\Sigma w\,d\sigma = 0$$

which means absence of expulsion of fluid, i.e. $p = 0$. Consequently, the function $\phi$ is known at the sections $z = 0$ and $z = l_1$; we consider these sections as the domain $\Sigma_2$. On the lateral surface we know the derivative

$\partial\phi/\partial n = \partial w/\partial t$, therefore we choose the wetted surface of the shell as our domain $\Sigma_1$.

In order to determine $\phi$ from Equation (1.6), we write down Green's function for the cylinder $0 \leqslant \alpha \leqslant l_1/R$, $r \leqslant R$, with the condition that $\partial G/\partial r = 0$ when $r = R$, and $G = 0$ when $\alpha = 0$ or $\alpha = l_1/R$:

$$G = \frac{8}{l_1} \sum_{p=1}^{\infty} \sum_{q=0}^{\infty}{}' \sin(\mu_p \alpha) \sin(\mu_p \alpha') g_{pq}(r, r') \cos(\beta - \beta') q \qquad (3.4)$$

where

$$\mu_p = \frac{p\pi R}{l_1}, \qquad g_{pq}(r, r') = \frac{I_q(\mu_p r'/R)}{I_q'(\mu_p)} \left[ I_q'(\mu_p) K_q\left(\mu_p \frac{r}{R}\right) - I_q\left(\mu_p \frac{r}{R}\right) K_q'(\mu_p) \right]$$

A prime at the sign of sum means that half the term of order zero should be taken.

Eliminating $\phi$ from the third equations of the system (1.7), we obtain

$$\frac{\partial^2 u}{\partial \alpha^2} + \frac{1-\nu^2}{2} \frac{\partial^2 u}{\partial \beta^2} + \frac{1+\nu}{2} \frac{\partial^2 v}{\partial \alpha \, \partial \beta} + \nu \frac{\partial w}{\partial \alpha} + \frac{1-\nu}{Eh} \left[ F_\alpha - m_0 \frac{\partial^2 u}{\partial t^2} \right] = 0$$

$$\frac{1+\nu}{2} \frac{\partial^2 u}{\partial \alpha \, \partial \beta} + \frac{\partial^2 v}{\partial \beta^2} + \frac{1-\nu}{2} \frac{\partial^2 v}{\partial \alpha^2} + \frac{\partial w}{\partial \beta} + \frac{1-\nu}{Eh} \left[ F_\beta - m_0 \frac{\partial^2 v}{\partial t^2} \right] = 0 \qquad (3.5)$$

$$\nu \frac{\partial u}{\partial \alpha} + \frac{\partial v}{\partial \beta} + c^2 \nabla^2 \nabla^2 w + \frac{1-\nu^2}{Eh} R^4 \left\{ \frac{1}{R^2} \left[ T_1^\circ \frac{\partial^2 w}{\partial \alpha^2} + T_2^\circ \left( \frac{\partial^2 w}{\partial \beta^2} + w \right) \right] + \right.$$

$$\left. + m_0 \frac{\partial^2 w}{\partial t^2} + \beta \iint_{\Sigma_1} G \frac{\partial^2 w}{\partial t^2} d\sigma \right\} = 0 \qquad (3.6)$$

The tangential components $m_0 \partial^2 u/\partial t^2$, $m_0 \partial^2 v/\partial t^2$ of the inertia forces and those of the reduced loading, namely the components $F_\alpha$ and $F_\beta$, are usually neglected in the theory of shells [2] in comparison with the normal components $m_0 \partial^2 w/\partial t^2$ and $F_n$, respectively.

If use is made of this simplification and if we introduce a new unknown function $\Phi(\alpha, \beta, t)$, connected to the old variables by the relations

$$u = c^2 \left( \frac{\partial^5 \Phi}{\partial \alpha^2} - \frac{\partial^5 \Phi}{\partial \alpha \, \partial \beta^4} \right) + \frac{\partial^3 \Phi}{\partial \alpha^2 \partial \beta} - \nu \frac{\partial^3 \Phi}{\partial \alpha^3}$$

$$v = 2c^2 \frac{\partial^3}{\partial \alpha^2 \partial \beta} \nabla^2 \Phi - (2+\nu) \frac{\partial^3 \Phi}{\partial \alpha^2 \partial \beta} + \frac{\partial^3 \Phi}{\partial \beta^3}, \qquad w = \nabla^2 \nabla^2 \Phi$$

then the first two equations will be fulfilled identically, while the third equation will serve for determination of $\Phi$:

$$(\nabla^2 + 1)^2 \nabla^2 \nabla^2 \Phi - (1-\nu) \frac{\partial^2}{\partial \alpha^2} \left( \frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \beta^2} \right) \nabla^2 \Phi + \frac{1-\nu^2}{c^2} \frac{\partial^4 \Phi}{\partial \alpha^4} +$$

$$+\frac{m_0R^4}{D}\frac{\partial^2}{\partial t^2}\nabla^2\nabla^2\Phi+\frac{R^4}{D}\left\{\left[T_1^\circ\frac{\partial^2}{\partial\alpha^2}+T_2^\circ\left(\frac{\partial^2}{\partial\beta^2}+1\right)\right]\nabla^2\nabla^2\Phi+\right.$$

$$\left.+\rho\iint\limits_{\Sigma_1}G\frac{\partial^2}{\partial t^2}\nabla^2\nabla^2\Phi d\sigma\right\}=0\qquad\left(D=\frac{Eh^3}{12(1-\nu^2)}\right)\tag{3.7}$$

In order to satisfy the boundary conditions

$$w=0,\quad\frac{\partial^2w}{\partial\alpha^2}=0,\quad v=0,\quad T_1=0\qquad\text{for}\quad\alpha=0\text{ and }\alpha=\frac{l}{R}\tag{3.8}$$

we prescribe for the function $\Phi$ the boundary conditions

$$\Phi=\frac{\partial^2\Phi}{\partial\alpha^2}=\frac{\partial^4\Phi}{\partial\alpha^4}=\frac{\partial^6\Phi}{\partial\alpha^6}=0\qquad\text{for}\quad\alpha=0\text{ and }\alpha=\frac{l}{R}\tag{3.9}$$

Satisfying these conditions we try to find the solution of the integro-differential equation (3.7) by the method of Galerkin in the form of a series

$$\Phi(\alpha,\beta,t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f_{mn}(t)\sin\lambda_m\alpha\cos n\beta\qquad\left(\lambda_m=\frac{m\pi R}{l}\right)\tag{3.10}$$

Substitution of (3.10) into Equation (3.7) gives

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\left[(\lambda_m^2+n^2+1)^2(\lambda_m^2+n^2)^2+(1-\nu)\lambda_m^2(\lambda_m^2-n^2)(\lambda_m^2+n^2)+\right.\right.$$

$$\left.+\frac{1-\nu}{c^2}\lambda_m^2\right]f_{mn}\sin\lambda_m\alpha\cos n\beta+\frac{m_0R^4}{D}(\lambda_m^2+n^2)\overset{4}{f_{mn}}\sin\lambda_m\alpha\cos n\beta-$$

$$-\frac{R^4}{D}[T_1^\circ\lambda_m^2+T_2^\circ(n^2-1)](\lambda_m^2+n^2)^2f_{mn}\sin\lambda_m\alpha\cos n\beta+$$

$$\left.+\rho\frac{R^4}{D}\iint\limits_{\Sigma_1}G(\lambda_m^2+n^2)f_{mn}''\sin\lambda_m\alpha'\cos n\beta'd\sigma'\right\}=0\tag{3.11}$$

Let us consider in more detail the last term of this equation.

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{8\rho R^4}{l_1D}\int\limits_0^{2\pi}\int\limits_0^{l_1/R}R^2g_{qp}(R_1R)(\lambda_m^2+n^2)^2f_{mn}\sin\mu_p\alpha\sin\mu_p\alpha'\times$$

$$\times\cos q(\beta-\beta')\cos n\beta'd\alpha'd\beta=$$

$$=\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}{\sum_{n=0}^{\infty}}'\frac{8\pi\rho R^6}{l_1D}g_{ph}(R_1R)(\lambda_m^2+n^2)f_{mn}''\cos n\beta\sin\mu_p\alpha\int\limits_0^{l/R}\sin\lambda_p\alpha'\sin\lambda_m\alpha'd\alpha'$$

$$=\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}{\sum_{n=0}^{\infty}}'(-1)^{p+1}\frac{8\pi\rho R^6}{l_1D}g_{ph}(R_1R)(\lambda_m^2+n^2)^2\sin\frac{m\pi l_1}{l}\frac{\lambda_m}{\mu_p-\lambda_m^2}\times$$

$$\times f_{mn}''\sin\mu_p\alpha\cos n\beta\tag{3.12}$$

Substituting this expression into Equation (3.11) and equating the coefficients of cos $n\beta$ to zero, we obtain the system of equations

$$\sum_{m=0}^{\infty}\left\{\left[(\lambda_m^2+n^2+1)^2(\lambda_m^2+n^2)^2+(1-\nu)\lambda_m^2(\lambda_m^4-n^4)+\frac{1-\nu^2}{c^2}\lambda_m^4\right]f_{mn}\sin\lambda_m\alpha-\right.$$
$$-\frac{R^2}{D}[T_1^\circ\lambda_m^2+T_2^\circ(n^2-1)](\lambda_m^2+n^2)^2 f_{mn}\sin\lambda_m\alpha+\frac{m_0R^4}{D}(\lambda_m^2+n^2)f_{mn}\sin\lambda_m\alpha+$$
$$\left.+\sum_{p=0}^{\infty}(-1)^p\frac{8\pi\rho R^6}{D}g_{pn}(R,R)\frac{(\lambda_m^2+n^2)^2\mu_p}{\lambda_m^2-\mu_p^2}\sin\frac{m\pi l_1}{l}f_{mn}''\sin\mu_p\alpha\right\}=0 \qquad (3.13)$$
$$(n=1,2,3,\ldots)$$

Multiplying the last equation by sin $\lambda_i\alpha$ and integrating between the limits 0 and $a = l/R$, we obtain for determination of $f_{mn}(t)$ the following infinite coupled system of equations:

$$I_{in}^\circ\frac{d^2f_{in}}{dt^2}+\sum_{m=0}^{\infty}I_{mni}\frac{d^2f_{mn}}{dt^2}+J_{in}^{(1)}f_{in}-\sum_{m=0}^{\infty}J_{mni}^{(2)}f_{mn}=0 \qquad (n,\ i=1,2,\ldots) \qquad (3.14)$$

Here, the notation is introduced

$$J_{in}^{(1)}=\frac{l}{2R}\left[(\lambda_i^2+n^2+1)^2(\lambda_i^2+n^2)+(1-\nu)\,\lambda_i^2(\lambda_i^4-n^4)+\frac{1-\nu^2}{c^2}\lambda_i^4\right]$$

$$J_{mni}^{(1)}=\frac{R}{D}\lambda_{im}^2(\lambda_m^2+n^2)^2\int_0^{l/R}T_1^\circ(\alpha,t)\sin\lambda_m\alpha\sin\lambda_i\alpha d\alpha+$$
$$+\frac{R^6}{D}(n^2-1)(\lambda_m^2+n^2)\int_1^{l/R}T_2^\circ(\alpha,t)\sin\lambda_m\alpha\sin d_i\alpha d\alpha$$

$$I_{mni}=\sum_{p=0}^{\infty}(-1)^{p+i}\frac{8\pi\rho R^6}{l_1D}g_{pn}(R_1R)\frac{(\lambda_m^2+n^2)\mu_p\lambda_i}{(\lambda_m^2+\mu_p^2)(\mu_p^2-\lambda_i^2)}\sin\frac{m\pi l_1}{l}\sin\frac{p\pi l}{l_1}$$

$$I_{in}^0=\frac{lm_0R^3}{2D}(\lambda_i^2+n^2)^2$$

The applicability of the method of Galerkin to the solution of the problem under consideration proves the convergence of the system of equations (3.14). This leads to the conclusion that an approximate solution of this infinite system can be obtained by using a truncated finite system of equations (3.14) with $n$, $i$ = 1, ..., $N$.

As a result we shall obtain a system of Hill's equations, and an investigation of this system will permit us to answer the question concerning the natural frequencies and critical forces for the system, consisting of shell and liquid, and to establish the region of dynamic stability in the space of the system parameters $n/R$, $l_1/R$, $l/R$, $m_0\ \rho$ for various external loadings.

**4.** Let us consider in greater detail the case when the liquid fills the entire inner space of the cylindrical shell. This corresponds to the case $l = l_1$. In the infinite sum $I_{mni}$ there remains the term $p = 1$ and, further, $I_{mni} = 0$ for all numbers $m \neq i$. For $m = i$ we obtain

$$I_{ini} = I_{ni} = \frac{2R^4 \rho l}{D} g_{in}(R, R)(\lambda_i^2 + h^2) \tag{4.1}$$

Furthermore, we replace the stress resultants $T_1^\circ(\alpha, t)$ and $T_2^\circ(\alpha, t)$ by the mean values

$$T_1^{\circ *}(t) = \frac{R}{l} \int_0^{l/R} T_1^\circ(\alpha, t)\, d\alpha, \qquad T_2^{\circ *} = \frac{R}{l} \int_0^{l/R} T_2^0(\alpha, t)\, d\alpha \tag{4.2}$$

Then the coefficients $J^{(2)}_{mni}$ vanish for all numbers $m \neq i$, while for $m = i$ they become

$$J_{in}^{(2)} = J_{ini}^{(2)} = \frac{R}{2Dl} \lambda_i^2 (\lambda_i^2 + n^2) T_1^{\circ *}(t) + \frac{R}{2Dl}(h^2 - 1)(\lambda_i^2 + h^2) T_2^{\circ *}(t) \tag{4.3}$$

By virtue of these simplifications the infinite coupled system (3.14) reduces to a system of ordinary Hill equations

$$(I_{in}^\circ + I_{in}) \frac{d^2 f_{in}}{dt^2} + [J_{in}^{(1)} - J_{in}^{(2)}(t)] f_{in}(t) = 0 \qquad (i, n = 1, 2, \ldots) \tag{4.4}$$

This system is conveniently written in the following form:

$$\frac{d^2 f_{in}}{dt^2} + \omega_{in}^2 \left(1 - \frac{T_1^{\circ *}(t)}{T_1^*} - \frac{T^{\circ *}(t)}{T_2^*}\right) f_{in}(t) = 0 \qquad (n, i = 1, 2, \ldots) \tag{4.5}$$

The quantities

$$\omega_{in}^2 = \frac{[(i\pi R/l)^2 + n^2 + 1]^2 D}{R^4 [m_0 + 4\pi\rho R g_{in}(R,R)]} + \\ + \frac{D(1-\nu)(i\pi R/l)^2 [(i\pi R/l)^4 - n^4] + (i\pi R/l)^4 (1-\nu^2)/l^2}{R^4 [m_0 + 4\pi R\rho g_{in}(R,R)] [(i\pi R/l)^2 + n^2]^2} \tag{4.6}$$

appearing in (4.5) are of the nature of natural frequencies of the shell; the quantity $4\pi\rho R g_{in}(R, R)$ represents the added mass due to the presence of the liquid in the shell. The quantities

$$T_1^* = \frac{\omega_{in}^2 [m_0 + 4\pi R\rho g_{in}(R, R)]}{(inR/l)^2}, \qquad T_2^* = \frac{\omega_{in}^2}{n^2 - 1} [m_0 + 4\pi\rho R g_{in}](R, R)]$$

represent the critical longitudinal and normal loadings; their physical meaning is that if the stress resultants $T_1^{\circ *}$ or $T_2^{\circ *}$ exceed those values the solution of Equations (4.5) will increase exponentially, which corresponds to the loss of stability of the shell.

The analysis of the dynamic stability of the shell can be carried out with the aid of a diagram of stability of Hill's equation (4.5). In this case it will be possible to indicate for any value of shell parameters whether the latter will be stable under the action of given forces.

It must be stated that the presence of liquid masses causes considerable decrease of the natural frequencies for the first harmonics of the bending mode. In some examples the ratio of decrease amounts to several hundred.

It is also essential that the critical values of the stress resultants for loss of static stability are not influenced by the presence of a liquid. It is, in analogy to the case of vibrations of a beam with a liquid [1], impossible to indicate an equivalent shell, although it is possible to indicate the equivalent mass of shell for each bending mode.

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